# Advanced solid-state physics

*List of key terms*

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## Part I. Band topology

ANOMALOUS VELOCITY: Contribution to the electron velocity caused by the band topology (Berry curvature), as opposed to the conventional group velocity, $\mathbf{v} = \partial\varepsilon/\partial\mathbf{k}$. Anomalous velocity results in the anomalous Hall effect. It can be particularly large in topological materials, such as magnetic Weyl semi-metals.

BAND STRUCTURE: Series of energy levels of the crystal measured as a function of the momentum, $\varepsilon_n = \varepsilon_n(\mathbf{k})$. In this lecture course, we mostly refer to the electronic band structure, but phonons, magnons, and any other quasiparticles can have their own band structures too.

BERRY PHASE: Additional phase acquired by the electron's wavefunction as the electron moves through a vector field. This field can be, for example, magnetic field in the case of the Aharonov-Bohm effect. In our solid-state context, the relevant field is given by the Berry curvature. The Berry phase is then the flux of the Berry curvature defined by the electron's trajectory in the $\mathbf{k}$-space.

- BERRY CONNECTION: Effective vector potential defined as $\mathbf{A}(\mathbf{k}) = i\langle u_\mathbf{k}|\boldsymbol{\nabla}|u_\mathbf{k}\rangle$ where $u_\mathbf{k}$ is the periodic part of the Bloch wavefunction, $\psi_\mathbf{k}(\mathbf{r}) = u_\mathbf{k}(\mathbf{r})\, e^{i\mathbf{kr}}$. Berry connection is not gauge-invariant, but its curl, the Berry curvature, is.
- BERRY CURVATURE: Mathematically defined as the curl of the Berry connection, $\boldsymbol{\Omega}(\mathbf{k}) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{k})$, Berry curvature is an effective magnetic field acting on the electron in the crystal. This effective field reflects the topology of the underlying electronic bands.

BULK-BOUNDARY CORRESPONDENCE: Bulk states of the crystal determine the nature of its surface states. More specifically, crystals with the topologically non-trivial band structure should feature characteristic surface states, such as chiral edge modes in a Chern insulator.

CHERN INSULATOR: 2D material characterized by a non-zero Chern number $C$ that also defines the number of its chiral edge modes.

- CHERN NUMBER: Topological invariant defined as the integral of the Berry curvature over a 2D manifold. In the solid-state context, this integral is calculated over a 2D Brillouin zone and yields an integer number (modulo $2\pi$) according to the Chern-Gauss-Bonnet theorem.

CHIRAL ANOMALY: Redistribution of electrons between two Weyl nodes with the opposite chirality when parallel electric and magnetic fields ($\mathbf{E} \parallel \mathbf{B}$) are applied to the crystal. The simplest manifestation of the chiral anomaly is the negative longitudinal magnetoresistance observed in Dirac and Weyl semi-metals.

CHIRAL EDGE MODE: A state localized on the edge of a 2D crystal. It allows electronic transport only in one direction, hence the name *chiral*. This chirality prevents electrons from scattering and facilitates ballistic (dissipationless) electric transport. Chiral edge modes occur in Chern insulators and 2D topological insulators.

FERMI ARC: Characteristic surface state of a Weyl semi-metal. It connects the projections of the bulk Weyl nodes.

HALDANE MODEL: Simplest model of a Chern insulator. It features a 2D hexagonal crystal (of graphene or BN type) with deliberately chosen second-neighbor hoppings that, in a certain range of parameters, lead to the band inversion (swap of the band character between the lower and upper bands) and result in the non-zero Chern number.

HALL EFFECT: Transverse component of the resistivity ($\rho_{xy}$) that appears in the presence of the magnetic field applied perpendicular to the current flow ($\mathbf{B} \parallel z$, $\boldsymbol{j} \parallel x$). Ordinary Hall effect is linear in $B$ ($R_H \sim B$) and used to probe carrier concentration. Many other forms of the Hall

effect exist too. They serve as fingerprints of the nontrivial band topology and exotic magnetic states in the crystal.

- Anomalous Hall effect (AHE): A contribution to the Hall resistivity, which is proportional to the magnetization and does not depend on the applied magnetic field: $\rho_{xy} = R_H B + R_a M$ where $R_H$ stands for the ordinary Hall effect and $R_a$ stands for the anomalous one. The AHE can arise from several mechanisms, the *extrinsic* one (spin-dependent or *skew scattering* on impurities) and the *intrinsic* one due to the Berry curvature of the underlying band structure.
- Fractional quantum Hall effect (FQHE): Quantized Hall resistance of the 2D electron gas that violates the usual sequence of $\rho_{xy} = h/(\nu e^2)$ plateaus with $\nu$ integer, as it would be expected from the filling of the Landau levels. Instead, one sees plateaus at the fractional values of $\nu$, such as $\frac{2}{3}$ or $\frac{4}{5}$ that arise from the so-called *composite fermions*, the electrons with a fraction of flux quantum attached to them. This trapping of flux by the electrons reduces the effective magnetic field and changes the Landau levels, such that their occupation may change with the fractional amount of charge. Fractional quantum Hall effect is a manifestation of *charge fractionalization* due to strong interactions between the electrons that are confined to the lowest Landau level of the system.
- Integer quantum Hall effect (IQHE): Quantized Hall resistance of the 2D electron gas, $\rho_{xy} = h/(\nu e^2)$. It arises from the quantized nature of the Landau levels in 2D and reflects the nontrivial topology of the QHE state. The QHE state also features chiral edge modes that are responsible for the electrical transport.
- Quantum anomalous Hall effect (QAHE): Quantized AHE in a Chern insulator, $\sigma_{xy} = C \times e^2/h$. Similar to IQHE, it appears in 2D only.
- Spin Hall effect (SHE): Transverse spin current induced by the electric current in the material. Contrary to the other Hall effects, SHE does not require any applied magnetic field. It arises from the spin-orbit coupling and occurs in materials with heavy elements as well as in topological insulators.

Kramers theorem: This theorem stipulates that the energy states of a system with time-reversal symmetry should be at least double-degenerate. In solid-state context, it means that energy bands feature a certain symmetry, $\varepsilon_\uparrow(\mathbf{k}) = \varepsilon_\downarrow(-\mathbf{k})$, and cross at special points known as time-reversal-invariant momenta ($\mathbf{k}$ is equivalent to $-\mathbf{k}$).

Landau levels: Series of energy levels of the crystal in the applied magnetic field. The splittings between the Landau levels are determined by the field strength (typically, the level energies are linear in $B$: $\varepsilon_n = \hbar\omega_c(n + \frac{1}{2})$ where $\omega_c = eB/m^*$ is the cyclotron frequency). The restructuring of the energy bands into Landau levels causes an oscillatory behavior of resistivity and other properties as a function of the field. In 2D, the Landau leves are discrete and lead to quantum Hall effect.

Magnetoresistance: Change in the electrical resistivity ($\rho_{xx}$) caused by the applied magnetic field. One distinguishes *longitudinal* ($\mathbf{B} \parallel x$) and *transverse* ($\mathbf{B} \parallel y$) magnetoresistance depending on the field direction with respect to the current. Magnetoresistance should not be confused with the Hall resistance, $\rho_{xy}$. Topological semi-metals feature huge and positive transverse magnetoresistance as well as negative longitudinal magnetoresistance (chiral anomaly).

Symmetry: A set of operations that leave the Hamiltonian of the system invariant. In solid-state context, three types of symmetry are usually considered:

- Inversion symmetry: Spatial inversion, $\mathbf{r} \to -\mathbf{r}$ and $\mathbf{k} \to -\mathbf{k}$ without changing the spin.
- Time-reversal symmetry: $\mathbf{r} \to \mathbf{r}$, $\mathbf{k} \to -\mathbf{k}$ and the reversal of spin. Systems with broken

time-reversal symmetry are magnetic, they typically feature a non-zero magnetization.

- Crystalline symmetry: Rotation axes and mirror planes defined by the crystal structure of the material. Such crystalline symmetries lead to additional constraints on the electronic structure. For example, they protect band crossings in Dirac and Weyl semi-metals.

Tight-binding model: Description of the band structure in terms of hopping parameters that represent effective interactions between atoms in the crystal: $\varepsilon(\mathbf{k}) = \sum_{\mathbf{R}} t_i e^{i\mathbf{kR}}$ where $\mathbf{R}$ are lattice vectors. Tight-binding models can be used for a deliberate design of the electronic states (we did so with the Haldane and Kane-Mele models) or to map the complex band structure of a real material onto a tractable minimum model.

Time-reversal-invariant momentum (TRIM): The value of $\mathbf{k}$, which is equivalent to $-\mathbf{k}$ by virtue of a reciprocal-lattice translation. Such a point should then feature double-degenerate bands (hence band crossings) in systems with time-reversal symmetry. The TRIM's typically include $\Gamma$, the corners of the Brillouin zone ($M$, $R$), and the midpoints between the centers of two adjacent Brillouin zones ($X$, $Y$, $Z$). The number of TRIM's is 4 in 2D and 8 in 3D.

Topological invariant: A parameter that stays invariant under adiabatic transformations and assigns the system to a certain topological class. In solid-state context, topological invariant describes the electronic structure of the material and does not change unless something major happens to the crystal (insulator turns into a metal, or the other way around). Two simplest topological invariants covered in this lecture course are the Chern number (Chern insulators, Weyl semi-metals) and $\mathbb{Z}_2$ index (topological insulators).

Topological insulator (TI): An insulator with metallic surface states. TI's are characterized by time-reversal symmetry. Kramers theorem then requires that states of the opposite spin cross at those points where $\mathbf{k}$ is equivalent to $-\mathbf{k}$ (time-reversal-invariant momenta) and form Dirac cones. TI's are characterized by the $\mathbb{Z}_2$ topological invariant. They feature characteristic spin textures and demonstrate large spin Hall effect as a ramification of the *spin-momentum locking*. TI's can occur in both 2D and 3D.

- Strong topological insulator: 3D topological insulator with the robust metallic surface states. This robustness comes from the fact the $\mathbb{Z}_2$ invariant of the underlying 3D band structure is 1.
- Weak topological insulator: 3D topological insulator that can be thought as a stack of 2D TI's. Its $\mathbb{Z}_2$ is 1 for some of the 2D cross-sections, yet 0 in 3D.

Topological semi-metals: 3D materials with linear band crossings in the vicinity of the Fermi level.

- Dirac semi-metal: A crystal with both inversion and time-reversal symmetries where all bands are double-degenerate (spin-up and spin-down), hence the band crossing involves four states. Additional crystalline symmetries are required in order to protect the band crossings. Dirac semi-metals do not have a simple topological invariant and do not feature protected surface states. However, they do show some characteristic properties, such as high electron mobility and huge transverse magnetoresistance. They also show the chiral anomaly, as the magnetic field splits the Dirac node into two Weyl nodes.
- Weyl semi-metals: A crystal where either of the inversion and time-reversal symmetries is broken, bands are non-degenerate, and each band crossing (Weyl node) involves only two states. Such Weyl nodes are chiral, serving as sources and sinks of the Berry curvature. A non-zero Chern number is then obtained by integrating Berry curvature on a plane separating the two Weyl nodes of opposite chirality. This Chern number is the topological invariant of a Weyl semi-metal. Surface states of Weyl semi-metals are Fermi arcs.

- NODAL-LINE SEMI-METAL: A crystal where band crossings form an infinite manifold, typically a ring or an open line. The high concentration of the Dirac/Weyl nodes leads to an enhanced response, for example in the anomalous Hall effect and spin Hall effect. Nodal-line semimetals feature drumhead surface states with large values of the surface DOS.

$\mathbb{Z}_2$ INVARIANT: Topological invariant of a topological insulator. It takes only two values, 0 and 1, that distinguish between the trivial (insulating surface) and topological (metallic surface) cases, respectively. In inversion-symmetric crystals, the $\mathbb{Z}_2$ invariant is obtained as a product of parities (eigenvalues of the inversion operator $\hat{I}$) at time-reversal-invariant momenta.

## Part II: Interacting electrons

BCS THEORY: Theory of conventional superconductivity and the most conventional theory of superconductors. It describes the superconducting state as a condensate of Cooper pairs stabilized by electron-phonon coupling. Main assumptions of this theory are: i) the momentum-independent pairing interaction ($V_{\mathbf{k}\mathbf{k}'} = \mathcal{V} = \text{const}$) leading to the $s$-wave superconductivity, and ii) the weak-coupling limit ($\mathcal{V}N(\varepsilon_F) \ll 1$). The central result of BCS is the explicit dependence of the superconducting gap and $T_c$ on three material-specific parameters: i) Debye frequency; ii) density of states at the Fermi level; iii) strength of the electron-phonon coupling.

BOGOLYUBOV QUASIPARTICLE: A distinct quasiparticle in superconductors formed by a combination of electrons and holes. The quasiparticle energies are given by $E_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2} \geq \Delta_{\mathbf{k}}$ where $\Delta_{\mathbf{k}}$ is the superconducting gap.

CHARGE-DENSITY WAVE (CDW): Weak structural deformation that (at least partially) gaps out the energy bands of a metal. The simplest example of a CDW is the *Peierls instability* of a metallic chain. CDW's often compete with superconductivity, because both states are usually triggered by the electron-phonon coupling.

CHARGE-TRANSFER ENERGY: The energy separation between the correlated $d$-bands of a transition-metal atom and ligand $p$-bands. Mott insulators are characterized by the large and positive charge-transfer energy ($\Delta_{\text{CT}}$). Materials with the low $\Delta_{\text{CT}}$ feature $p$-states near the Fermi level and thus deviate from the simple Mott-Hubbard scenario.

COOPER PAIR: A bound state of two electrons that get attracted to each other by virtue of the electron-phonon coupling. Cooper pairs are bosons and usually singlets ($S = 0$). Their condensate is the superconducting state.

DENSITY FUNCTIONAL THEORY (DFT): A major computational approach in solid-state physics that relies on the fact that total energy of the system is a functional of its electron density. While the exact form of this functional is unknown, many realistic approximations exist. These approximations lead to the one-electron Kohn-Sham equations that describe the behavior of an electron in an effective periodic potential and comply with the Bloch theorem. As long as a given material can be described by DFT, it is classified as weakly correlated, because band theory describes it just well.

ELECTRONIC CORRELATIONS: Electron-electron interactions that drive many-body physics beyond the conventional description in terms of the band theory. Materials are called *strongly correlated* if they behave qualitatively different from the predictions of the band theory. One major example are Mott insulators that would be metallic in the absence of correlations.

EXCHANGE INTERACTION: An interaction between magnetic moments that appears as a ramification of the electron-electron repulsion. Magnetic configuration that minimizes the repulsion is favored over any other magnetic configuration where the repulsion would be stronger. On

a microscopic level, exchange interaction can be obtained by a perturbative treatment of the Hubbard model in the insulating limit ($t \ll U$).

FRACTIONALIZATION: A general phenomenon describing quasiparticles that can not be formed from natural constituents of the system. Examples of fractionalization include the breakdown of magnons ($S = 1$ bosons) into *spinons* ($S = \frac{1}{2}$ fermions) in a quantum spin chain, and the reduction of electron charge within the fractional quantum Hall states.

HUBBARD MODEL: The quintessential model of strongly correlated electrons. It describes the behavior of electrons that can move through the crystal via the hoppings $t$ subject to the strong on-site Coulomb repulsion $U$. Such Coulomb repulsion will typically arise in the strongly localized $d$- and $f$-shells, yet it is also relevant to all systems with weakly dispersive (flat) bands where even weak electron-electron interactions can lead to manifestations of strongly correlated behavior.

GOLDSTONE MODE: A gapless bosonic mode that appears as an elementary excitation when continuous symmetry of the system is broken. Examples include magnon modes in ordered (anti)ferromagnets and phonons in crystals (where periodic crystal is seen as a symmetry-broken state of a liquid).

LANDAU FERMI LIQUID: Theoretical framework for describing metals with electron-electron interactions. It rests upon an assumption that interacting electrons can be replaced by weakly interacting quasiparticles. Consequently, correlated metals behave similar to simple (uncorrelated) metals, but with the renormalized Sommerfeld coefficient, compressibility, and Drude spectral weight. Quadratic temperature dependence of the resistivity, $\rho(T) = \rho_0 + AT^2$, is a fingerprint of the spin-liquid behavior.

LANDAU QUASIPARTICLE: A cornerstone of Landau Fermi liquid theory: electrons "dressed" by their interactions with other electrons. To a first approximation, these quasiparticles are electrons with the renormalized (effective) mass $m^*$ that leads to a renormalization of the density of states at the Fermi level.

MAGNON: Elementary excitation of a magnetically ordered state. The classical analog of magnon is a *spin wave* formed by magnetic moments that are weakly tilted with respect to their equilibrium positions. Magnons are somewhat similar to phonons, but they may have a different type of dispersion, most notably, $\hbar\omega_{\mathbf{k}} \sim k^2$ in ferromagnets vs. the phonon-like $\hbar\omega_{\mathbf{k}} \sim k$ in antiferromagnets.

MOTT INSULATOR: The insulating state that forms when a half-filled band is split by electronic correlations. Mott insulators appear in the Hubbard model at half-filling and $t \ll U$. They are fundamentally different from *band insulators*, which arise from the complete filling of the energy bands of uncorrelated electrons.

SPIN WAVE: see Magnon.

- LINEAR SPIN-WAVE THEORY: The description of low-energy excitations of a magnetically ordered state in terms of non-interacting magnons. Mathematically, it is obtained by neglecting higher-order terms in the spin Hamiltonian. Linear spin-wave theory fails when multiple magnons are excited. It also fails in systems with strong quantum fluctuations, especially in low-dimensional magnets.

SPINON: Fractionalized magnetic excitations in low-dimensional magnets, such as quantum spin chains. In contrast to spin-1 magnons, spinons carry spin-$\frac{1}{2}$.

STRANGE METAL: A correlated metallic state characterized by the linear temperature dependence of the resistivity down to $T = 0$ (the linear dependence is also expected in simple metals, but only at elevated temperatures where phonon scattering becomes dominant). Strange metals

are often precursors of unconventional superconductors.

SUPERCONDUCTING GAP: The energy gap in the spectrum of elementary excitations of a superconductor. These excitations are Bogolyubov quasiparticles (mixtures of electrons and holes). In a simple language, the opening of this gap is related to the electron-phonon coupling that involves the states near the Fermi level. Such states are then unavailable for excitations, because occupying them would lead to a loss in the electron-phonon coupling energy.

SUPEREXCHANGE: An example of exchange interaction that does not involve any direct orbital overlap between interacting atoms. Superexchange is mediated by nonmagnetic ligand atoms, yet on the theoretical level it falls into the same category as the usual exchange interaction because it can be obtained by a perturbative treatment of the Hubbard model at the half-filling and $t \ll U$. The Hubbard model is restricted to the correlated states, whereas all ligand states are simply integrated out and implicitly included into the hoppings parameters.